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WHICH RECONSTRUCTION RESULTS ARE SIGNIFICANT?(U)
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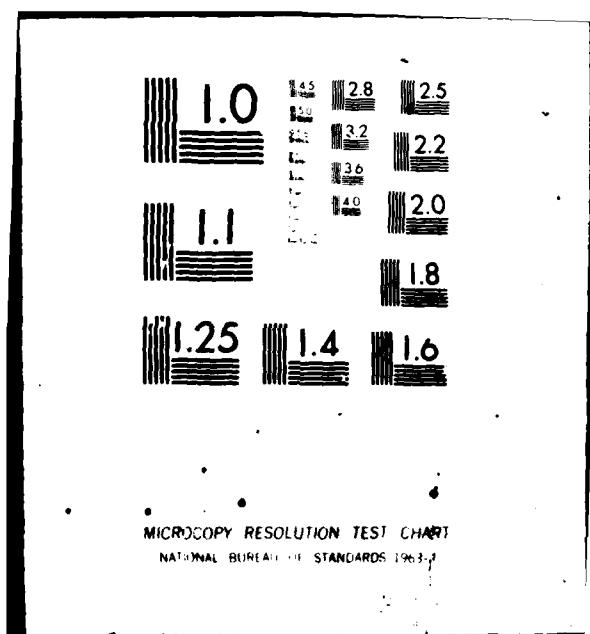
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JUN 18 1980

Technical Report 19

Presented at the Fourth International Conference
on the Theory and Applications of Graphs

Western Michigan University
Kalamazoo, Michigan

May 1980

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INTRODUCTION

One of the most famous unsolved problems in graph theory is the Reconstruction Conjecture, formulated nearly forty years ago by P. J. Kelly and/or S. M. Ulam. Well over one hundred papers have been written on the conjecture, including excellent survey articles by Bondy and Hemminger [1] and by Nash-Williams [10].

Intuitively, the conjecture states that every graph G with $p \geq 3$ vertices is uniquely determined by, or can be uniquely reconstructed from, its vertex-deleted subgraphs $G - \{v_i\}$, $1 \leq i \leq p$. A more formal statement of the conjecture, and one more useful for our purposes, uses the concept of a hypomorphism, introduced by Nash-Williams [10]. A hypomorphism from a graph G to a graph H is a bijection σ from the vertex set of G to the vertex set of H such that $G - \{v\}$ is isomorphic to $H - \{\sigma(v)\}$ for each vertex v of G . Two isomorphic graphs are clearly hypomorphic; the Reconstruction Conjecture asserts that the converse is true.

RECONSTRUCTION CONJECTURE: If two graphs G and H with $p \geq 3$ vertices are hypomorphic, then they are isomorphic.

Over the years, a rather large number of partial results and special cases has fueled a growing belief that the conjecture is true. Some doubt was cast

*Research supported by the Office of Naval Research under contract N00014-76-C-0673, NR044-459.

on this belief, though, by the recent discovery of infinite families of non-reconstructible digraphs, that is, digraphs which are hypomorphic but not isomorphic [15], [16]. In this paper we examine many of the reconstruction results that have been proved, searching for those that still can be viewed as providing strong evidence in support of the conjecture. In an attempt to add some precision to this rather vague concept, we call a reconstruction result significant if the analogous statement for digraphs is false. Clearly the "significant" results are those whose value as evidence is not weakened by the existence of non-reconstructible digraphs; any proof of the Reconstruction Conjecture must depend on one or more "significant" results in an essential way, since otherwise the proof would hold for digraphs as well.

Two comments are in order: first, many "non-significant" results are quite interesting and important. They should not be considered inferior or of no value simply because they fail to satisfy a somewhat artificial definition. Second, we will not consider reconstruction results concerning infinite graphs, matrices, relations, matroids, geometries, and other vaguely graphical structures. Regardless of whether or not they satisfy the definition for significance, they do not contribute to the spirit behind it.

SOME ELEMENTARY RESULTS

The first reconstruction result one discovers is that the number of vertices and the number of edges of a graph can be reconstructed from its vertex-deleted subgraphs. Not surprisingly, this is non-significant; the same proofs work for digraphs as well. Next, one discovers that the degree sequence of a graph can be determined. This, too, is non-significant, although not obviously so. Harary and Palmer [4] proved that the score sequence of a tournament can be determined, for $p \geq 5$, and Manvel [8] extended this to the degree-pair

sequence of general digraphs, again for $p \geq 5$. Knowing the degree sequence, it is easy to see that regular graphs are reconstructible. The same proof works for digraphs, however, so again we have a non-significant result.

Further results along these lines usually require tools such as the following:

Kelly's lemma [5]: For any graphs F and G where F has fewer vertices, the number $s(F, G)$ of subgraphs of G isomorphic to F is reconstructible.

Another useful tool, the counting theorem of Greenwell and Hemminger [2], is a generalization of Kelly's lemma. Neither of these tools is significant, as the same proofs work for digraphs. It is not surprising, then, to find that the consequences of these tools are not significant either. Thus, connectedness of a graph can be determined, but Manvel [9] has shown that connectedness of digraphs (either disconnected, weakly connected, unilaterally connected or strongly connected) can also be determined, for $p \geq 5$.

A related result is that disconnected graphs are reconstructible. The significance of this result depends on which type of digraph connectedness one considers. If one chooses weak connectivity, then the result is non-significant, since non-weakly connected digraphs are reconstructible for $p \geq 3$. Also, Harary and Palmer [4] have proved that non-strong tournaments are reconstructible for $p \geq 5$. However, we shall see later that there are arbitrarily large non-strong digraphs that are not reconstructible. Thus the reconstruction of disconnected graphs can be considered a significant result if one considers connectedness to mean strong connectivity.

Another consequence of Kelly's lemma is that trees are reconstructible. Harary and Palmer [3] have observed, however, that oriented trees are also

reconstructible, for $p \geq 4$, so this result also fails to be significant. Similar results include the reconstruction of unicyclic graphs, cacti, and other families of graphs with relatively few edges. Although the details have not been checked, it seems doubtful that any of these results are significant either.

TWO DEEPER RESULTS

There are at least two other results, both quite fascinating, that are sometimes presented as evidence in favor of the Reconstruction Conjecture. The first is the result of Lovász [6] on reconstruction from edge-deleted subgraphs.

Lovász's Theorem: Any graph G with more than $p(p - 1)/4$ edges is reconstructible from its edge-deleted subgraphs.

The proof utilizes an ingenious application of the principle of inclusion and exclusion. Elegant as it is, however, the result is not significant. The same proof works for digraphs, with the bound raised to $p(p - 1)/2$. Müller [12] extended the method of Lovász, bringing the bound down to $p \log(p)$, and this proof is also valid for digraphs. This incidentally provides an alternative proof of the result of Harary and Palmer [4] that tournaments are edge-reconstructible.

The other result we consider in this section is the probabilistic theorem of Müller [11].

Müller's Theorem: For every $\epsilon > 0$, almost all graphs on n vertices are reconstructible from their collections of induced $n(1 + \epsilon)/2$ - vertex subgraphs.

Although the details are rather tedious, this result is a consequence of showing that for almost all graphs, the induced $n(1 + \epsilon)/2$ - vertex subgraphs are all non-isomorphic and asymmetric. A bit more tedium shows that almost all

digraphs are reconstructible from their induced $n(1 + \epsilon)/4$ - vertex subgraphs, so this result of Miller is not significant. In fact we have the somewhat surprising fact that most digraphs are reconstructible from even less information than most graphs seem to require.

SOME SIGNIFICANT RESULTS

At this point one might well suspect that no reconstruction results are significant. This is not true, and we will present two that are. In order to discuss them adequately, we first present six families of non-reconstructible digraphs. These are illustrated by their adjacency matrices in Figure 1 for $p = 6$, where each digraph is hypomorphic, but not isomorphic, to its starred companion. The entries in the two blocks on the diagonal of each matrix are defined by $a_{ij} = 1$ iff $\text{odd}(j - i) \equiv 1 \pmod{4}$, where $\text{odd}(k)$ is the integer that results from dividing k by the largest possible power of 2. The off-diagonal blocks are filled in as shown to form the various examples. These digraphs, which exist for every order $p = 2^i + 2^j$, with $i > j \geq 0$, are described more fully in [16].

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$$A = \begin{array}{|ccccc|} \hline & 0 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$A^* = \begin{array}{cc|cc} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$B = \begin{array}{cc|cc} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}$$

$$B^* = \begin{array}{|ccccc|c|} \hline & 0 & 1 & 1 & 0 & | & 0 & 0 \\ & 0 & 0 & 1 & 1 & | & 0 & 0 \\ & 0 & 0 & 0 & 1 & | & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 & | & 0 & 0 \\ & 0 & 1 & 0 & 1 & | & 0 & 1 \\ & 1 & 0 & 1 & 0 & | & 0 & 0 \\ \hline \end{array}$$

$$C = \begin{array}{r} \begin{array}{ccccc|cc} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array} \end{array}$$

$$C^* = \begin{array}{cc|cc} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{array}$$

$$D = \begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \end{array}$$

$$D^* = \begin{array}{cc|cc} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}$$

$$E = \begin{array}{c|ccccc} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}$$

$$E^* = \begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}$$

$$F = \begin{array}{r} \begin{array}{cccc|c} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \end{array}$$

$$F^* = \begin{array}{r|l} \begin{array}{cccccc} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} & \end{array}$$

$$\begin{array}{r|l} \hline 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

Figure 1. Non-reconstructible Digraphs.

Our first significant result is unexpectedly elementary.

First Significant Theorem. If σ is a hypomorphism from graph G to G^* , then for each vertex v of G , the degree $d(v) = d(\sigma(v))$.

Note that this says more than the fact that hypomorphic graphs have the same degree sequence (a non-significant result). The significant aspect is that the degree of the missing vertex can be determined for each subgraph. This correspondence does not work for digraphs. The standard hypomorphism for the digraph pairs of Figure 1 is $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 2$, $\sigma(4) = 1$, $\sigma(5) = 6$, and $\sigma(6) = 5$. In each case, a vertex with degree pair (d_1, d_2) in one digraph is always mapped onto a vertex with degree pair $(d_1 - 1, d_2 + 1)$ or $(d_1 + 1, d_2 - 1)$ in the other digraph. This non-correspondence happens for all digraphs in these six families except when p is odd, when the degree pair of v equals that of $\sigma(v)$ for precisely one vertex v . Similar behavior is displayed by other non-reconstructible digraphs. In fact, the above theorem seems to be highly significant, in that it is violated by all known pairs of digraphs that are hypomorphic but not isomorphic.

The other significant result is due to Tutte [17].

Tutte's Theorem: The characteristic polynomial of a graph can be reconstructed. Equivalently, two hypomorphic graphs must have the same characteristic polynomial.

Several points should be noted concerning this theorem:

1. The derivative of the characteristic polynomial is the sum of the characteristic polynomials of the vertex-deleted subgraphs. Thus the characteristic polynomials of hypomorphic graphs can differ only in the constant term, so we can restrict our attention to the determinants of graphs. This observation holds for digraphs as well (hence it is non-significant).

2. A digraph and its converse are cospectral. Thus if a digraph is hypomorphic to its converse it will obey Tutte's theorem. This is the case for the pair C and C^* when p is odd, and for the pair D and D^* when p is even.

3. It was observed by Pouzet [13] that Tutte's theorem is equivalent to showing that the number of Hamiltonian circuits in a graph can be reconstructed, or that hypomorphic graphs have the same number of Hamiltonian circuits. In fact, Tutte used this equivalence in his proof. In our examples, the digraphs A , A^* , B , and B^* all have zero Hamiltonian circuits (they are non-strong) and thus satisfy Tutte's theorem.

4. The idiosyncratic polynomial of a graph G is the characteristic polynomial the matrix obtained by replacing each non-diagonal zero in the adjacency matrix of G with the indeterminant α . Tutte's theorem also holds for the idiosyncratic polynomial (and in particular the idiosyncratic determinant) of a graph. The digraphs A , A^* , B , and B^* violate this stronger result, since their complements (namely F^* , F , E^* , and E) have determinants differing from those of their mates.

Tables 1 and 2 list the known values of the determinants of our examples. Several patterns are obvious, such as the signs of the determinants, and the correspondence between D and D^* when $p = 2^k + 1$ and C^* and C for $p = 2^k + 2$. Most striking, however, is the fact that for each pair that is not forced to have identical determinants by either 2. or 3. above, the determinants differ by precisely one. We have so far been unsuccessful in proving that these patterns continue, but the evidence is fairly impressive. As with the other significant result, Tutte's theorem seems highly significant: it appears that all known pairs of digraphs that are hypomorphic but not isomorphic or mutually converse have different idiosyncratic determinants.

p	$A, A^*,$ $B, \text{ and } B^*$	$C \text{ and } C^*$	D	D^*	$E \text{ and } E^*$	$F \text{ and } F^*$
3	0	0	0	1	1	0
5	0	-1	3	2	1	2
9	0	-38	55	54	16	17
17	0	-20627	24307	24306	3679	3680
33	0	-4,393,732,430	4,753,524,979	4,753,524,978	359,792,548	359,792,549

Table 1. Determinants of Digraphs, p odd.

P	A, A*, B and B*		C	C*	D and D*	E and F	E* and F*
	0	2					
6							
10	0	54	55	-71	-17	-16	
12	14	211	210	-232	-35	-36	
18	0	24,306	24,307	-27986	-3680	-3679	
20	3401	83,680	83,679	-87,916	-7637	-7638	
24	47,614	1,129,371	1,129,370	-1,155,566	-73,809	-73,810	
34	0	4,753,524,978	4,753,524,979	-5,113,317,527	-359,792,549	-359,792,548	
36	346,176,266	15,326,336,299	15,326,336,298	-15,713,361,412	-733,201,379	-733,201,380	
40	4,846,467,724	189,057,025,701	189,057,025,700	-191,237,244,353	-7,026,686,376	-7,026,686,377	

Table 2. Determinants of Digraphs, p even.

CONCLUSIONS

The consensus at the First International Conference in 1968 was that the Reconstruction Conjecture was almost certainly true. The presentation of Manvel [7] conveys the mood at that time. Evidence suggested that the vertex-deleted subgraphs held more than enough information to specify the graph. This led to stronger conjectures: Manvel's Conjecture that a proper sub-collection of vertex-deleted subgraphs would suffice for large p ; Kelly's Conjecture that multi-vertex-deleted subgraphs would suffice, again for large p ; and Harary's Conjecture, that a graph can be reconstructed from its set of non-isomorphic subgraphs (without being given the multiplicity of each).

The examination of significant theorems leads naturally to two more conjectures. Each is stronger than the Reconstruction Conjecture for graphs, but weaker than that for digraphs. The first has been proposed independently by S. Ramachandran [14].

Conjecture 1. Every digraph is uniquely determined by its collection of vertex-deleted subdigraphs, together with the degree pair of each corresponding vertex.

Conjecture 2. Every digraph is determined up to conversity by its vertex-deleted subdigraphs together with its idiosyncratic determinant.

Is the Reconstruction Conjecture true? The optimism of twelve years ago is probably still warranted. As we have seen, all known non-reconstructible digraph pairs enjoy rather special properties that graphs cannot possess. In the broad sense of providing evidence either for or against the Reconstruction Conjecture, the non-reconstructible digraphs now known should not be considered particularly significant.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) College of William and Mary Department of Mathematics and Computer Science Williamsburg, VA 23185		20. REPORT SECURITY CLASSIFICATION Unclassified
21. GROUP		
3. REPORT TITLE 6. Which Reconstruction Results are Significant?		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report 19, May 1980		
5. AUTHOR(S) (First name, middle initial, last name) 10. Paul K. Stockmeyer		
11. REPORT DATE May 1980		12. TOTAL NO. OF PAGES 13
13. CONTRACT OR GRANT NO. N80014-76-C-0673		14. NO. OF REFS 17
15. PROJECT NO. 16. 12/16/1980 17. 9		
18. ORIGINATOR'S REPORT NUMBER(S) Technical Report 19		
19. REPORTING REPORT NO(S) (Any other numbers that may be assigned this report)		
20. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited.		
21. SUPPLEMENTARY NOTES		22. SPONSORING MILITARY ACTIVITY Mathematics Program Office of Naval Research Arlington, VA 22217
23. ABSTRACT It has been known for several years that the graph theory Reconstruction Conjecture is not true for directed graphs. In fact, several infinite families of non-reconstructable digraphs have been discovered. With this in mind, we call a graphical reconstruction result <u>significant</u> if the analogous statement for digraphs is false. Clearly any proof of the R. C. will depend on one or more "significant" results, since otherwise the proof would hold for digraphs as well. In this paper we survey the reconstruction literature, examining theorems for significance. The answers are often counterintuitive.		
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S/N. 0101-807-6801

Security Classification

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Graph Reconstruction Eigenvalues Digraphs						